The development of techniques (notably model checking) for the computer-aided verification of computing systems has been a truly successful application of logic to computer science.

2007 ACM Turing Award (Clarke, Emerson and Sifakis) “for their rôle in developing model checking into a highly effective verification technology, widely adopted in hardware and software industries”.

What is Model Checking?

Problem:

Given a system \( \text{Sys} \) (e.g. an OS) and a correctness property \( \text{Spec} \) (e.g. deadlock freedom), does \( \text{Sys} \) satisfy \( \text{Spec} \)?

The model checking approach:

1. Find an abstract model \( M \) of the system \( \text{Sys} \).
2. Describe the property \( \text{Spec} \) as a formula \( \phi \) of a (decidable) logic.
3. Exhaustively check if \( \phi \) is violated by \( M \).
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Verification of Higher-Order Programs

In the past two decades, there have been significant advances in the theory and engineering of scalable software model checkers (especially for first-order imperative programs such as C). E.g. SLAM, BLAST, CMBC.

- These techniques are much less useful for higher-order programs.
- Yet higher-order features (e.g. lambdas, streams) are already standard in today’s leading languages: Java8, C++11, C#5.0, Python, Scala, etc.

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Verifying higher-order functional programs: 2 standard approaches

1. Type-based program analysis. E.g. type-and-effect, qualifier, linear
   - sound, scalable but often imprecise

2. Theorem proving and dependent types. E.g. Coq, Agda
   - accurate, typically requires human intervention; does not scale well

We present an approach to verifying higher-order programs via higher-order model checking.
Higher-Order Model Checking is the model checking of infinite structures, such as trees, that are defined by recursion schemes (equivalently $\lambda Y$-calculus) and related families of higher-order generators.

This tutorial has four parts:

1. Introduction (Ong)
2. Applications to Program Verification (Kobayashi)
3. Type Systems and Algorithms for Higher-Order Model Checking (Kobayashi)
4. Advanced Topics (Ong)
**Simple Types (Church JSL 1940)**

**Types**

\[ A ::= o \mid (A \rightarrow B) \]

\(o\) is the type of trees.

**Order** of a type: measures “nestedness” on LHS of \(\rightarrow\).

\[
\text{order}(o) := 0 \\
\text{order}(A \rightarrow B) := \max(\text{order}(A) + 1, \text{order}(B))
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\]

**Examples**

1. \(\mathbb{N} \rightarrow \mathbb{N}\) and \(\mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N})\) both have order 1;
2. \((\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}\) has order 2.

**Notation**

\(e : A\) means “expression \(e\) has type \(A\)”. 
Higher-Order Recursion Schemes (HORS)

(Park 68, de Roever 72, Nivat 72, Nivat-Courcelle 78, Damm 82, ...)

HORS are grammars for trees (and tree languages).

Order-$n$ recursion schemes over $\Sigma = \text{programs of the order-}n \text{ fragment of } \lambda \rightarrow Y\text{-calculus} \text{ (i.e. simply-typed } \lambda\text{-calculus } + Y + \text{ order-1 } \Sigma\text{-symbols).}$
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Order-\(n\) recursion schemes over \(\Sigma\) = programs of the order-\(n\) fragment of \(\lambda \rightarrow \text{Y}\)-calculus (i.e. simply-typed \(\lambda\)-calculus + \(\text{Y}\) + order-1 \(\Sigma\)-symbols).

Concretely, a HORS is a finite set of simply-typed (higher-order) functions, defined by mutual recursion over \(\Sigma\), with a distinguished start function \(S\) of ground type.

Example (order 1). \(\Sigma = \{ f : o \rightarrow (o \rightarrow o), \ g : o \rightarrow o, \ a : o \}. \)

\[
\begin{align*}
G : \quad & S \rightarrow Fa \\
& F x \rightarrow (f x)(F (g x))
\end{align*}
\]
Example (order 1)

\[ \Sigma = \{ f : o \to (o \to o), \ g : o \to o, \ a : o \}. \]

\[ G : \begin{cases} 
S & \to \ F a \\
F x & \to \ (f x) (F (g x)) 
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\[
S \quad \to \\
\quad \quad \quad \to \ (f \ a) (F (g \ a)) \\
\quad \quad \quad \to \ (f \ a) (f (g \ a) (F (g (g \ a)))) \\
\quad \quad \quad \to \ \ldots
\]

The tree generated, \( [G] \), is the abstract syntax tree underlying \( (f \ a) (f (g \ a) (f (g (g \ a)))(\cdots)) \).

Many equivalent ways of defining \( [G] \) (as least fixpoint, least solution, initial algebra, etc.).
E.g. Consider properties of nodes of $[G]$:

- $\varphi = \text{“Infinitely many } f\text{-nodes are reachable”}$.
- $\psi = \text{“Only finitely many } g\text{-nodes are reachable”}$.

Every node of the tree satisfies $\varphi \lor \psi$.

Monadic second-order logic (MSO) is an expressive logic that can describe correctness properties such as $\varphi \lor \psi$. 
**E.g.** Consider properties of nodes of $\llbracket G \rrbracket$:
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**Monadic second-order logic (MSO)** is an expressive logic that can describe correctness properties such as $\varphi \lor \psi$.

**MSO Model-Checking Problem for Trees generated by HORS**
- **INSTANCE:** An order-$n$ recursion scheme $G$, and an MSO formula $\varphi$
- **QUESTION:** Does the $\Sigma$-labelled tree $\llbracket G \rrbracket$ satisfy $\varphi$?

**QUESTION:** Is the above problem decidable?
Theorem (O. LICS06)

For $n \geq 0$, the alternating parity tree automaton (APT) model-checking problem for order-$n$ recursion schemes is $n$-EXPTIME complete. Hence the MSO model checking problem is decidable.
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For \( n \geq 0 \), the alternating parity tree automaton (APT) model-checking problem for order-\( n \) recursion schemes is \( n \)-EXPTIME complete. Hence the MSO model checking problem is decidable.

Proofs of Decidability of HOMC / Models of Higher-Order Computation

1. Game semantics (O. LICS06)
2. Collapsible pushdown automata (Hague, Murawski, O. & Serre LICS08)
3. Intersection types (Kobayashi & O. LICS09)
4. Krivine machine (Salvati & Walukiewicz ICALP11)
Outline of Part 4

   - Properties of the Maslov (= Higher-order Pushdown) Hierarchy of Word Languages
   - Computing Downwards Closure of Higher-order Pushdown Languages

2. Model Checking Higher-type Böhm Trees
   - Challenge of Compositional Higher-order Model Checking
   - Automata-Logic-Games Correspondence for Higher-type Computation

3. Some Open Problems
Outline

1 Higher-Order Pushdown Automata: A Model of Higher-order Computation
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Order-2 pushdown automata

A 1-stack is an ordinary stack. A 2-stack (resp. \((n + 1)\)-stack) is a stack of 1-stacks (resp. \(n\)-stack).
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Operations on 2-stacks: $s_i$ ranges over 1-stacks.

\[
\begin{align*}
push_2 & : \ [s_1 \cdots s_{i-1} \underbrace{[\gamma_1 \cdots \gamma_n]}] \quad \mapsto \quad [s_1 \cdots s_{i-1} s_i s_i] \\
pop_2 & : \ [s_1 \cdots s_{i-1} [\gamma_1 \cdots \gamma_n]] \quad \mapsto \quad [s_1 \cdots s_{i-1}] \\
push_1 \gamma & : \ [s_1 \cdots s_{i-1} [\gamma_1 \cdots \gamma_n]] \quad \mapsto \quad [s_1 \cdots s_{i-1} [\gamma_1 \cdots \gamma_n \gamma]] \\
pop_1 & : \ [s_1 \cdots s_{i-1} [\gamma_1 \cdots \gamma_n \gamma_{n+1}]] \quad \mapsto \quad [s_1 \cdots s_{i-1} [\gamma_1 \cdots \gamma_n]]
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\text{push}_1 \gamma & : \ [s_1 \ldots s_{i-1} \ [\gamma_1 \ldots \gamma_n]] \quad \mapsto \quad [s_1 \ldots s_{i-1} \ [\gamma_1 \ldots \gamma_n \gamma]] \\
\text{pop}_1 & : \ [s_1 \ldots s_{i-1} \ [\gamma_1 \ldots \gamma_n \gamma_{n+1}]] \quad \mapsto \quad [s_1 \ldots s_{i-1} \ [\gamma_1 \ldots \gamma_n]]
\end{align*}
\]

Idea extends to all finite orders: an order-\(n\) PDA has an order-\(n\) stack, and has \(\text{push}_i\) and \(\text{pop}_i\) for each \(1 \leq i \leq n\).
Example: \( L := \{ a^n b^n c^n : n \geq 0 \} \) is recognisable by an order-2 PDA

\( L \) is not context free—thanks to the “\( uvwxy \) Lemma”.

---

Kobayashi, Ong
Higher-Order Model Checking
18 Jan 16, POPL16 Tutorial
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```
\begin{align*}
q_1 \, &\Downarrow a \rightarrow q_1 \, \Downarrow a \rightarrow q_1 \, \Downarrow b \rightarrow q_2 \, \Downarrow b \\
&\Downarrow c \leftarrow c \leftarrow c \leftarrow \Downarrow c \leftarrow c
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$q_1 \quad [\quad] \quad \xrightarrow{a} \quad q_1 \quad [\quad] \quad [z] \quad \xrightarrow{a} \quad q_1 \quad [\quad] \quad [z] \quad [z \ z]$

$q_2 \quad [\quad] \quad [z] \quad [z] \quad \xrightarrow{b} \quad q_3 \quad [\quad] \quad [z] \quad [\quad] \quad \xrightarrow{b} \quad q_2 \quad [\quad] \quad [z] \quad [\quad]$

\[ q_3 \quad [\quad] \quad \leftarrow c \quad q_3 \quad [\quad] \quad [z] \quad \leftarrow c \quad q_2 \quad [\quad] \quad [z] \quad [\quad] \]

- $\rightarrow push_2 \ ; \ push_1 z$
- $z \rightarrow pop_1$
- $z \rightarrow pop_2$

'read $a$'

'read $b$'

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Some properties of the Maslov Hierarchy (Maslov 74, 76)

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5. The **emptiness problem** of nondeterministic order-$k$ PDA is $(k - 1)$-EXPTIME complete. (Engelfriet '81)

A recent breakthrough

**Theorem (Inaba + Maneth FSTTCS08)**

*All languages of the Maslov Hierarchy are context-sensitive. So Maslov Hierarchy refines Chomsky Hierarchy.*
Theorem (Equi-expressivity)

For each $n \geq 0$, the three formalisms

1. order-$n$ pushdown automata (Maslov 76)
2. order-$n$ safe recursion schemes (Damm 82, Damm + Goerdt 86)
3. order-$n$ indexed grammars (Maslov 76)

generate the same class of word languages.
Downward closure of $\mathcal{L}$, $\downarrow(\mathcal{L})$, is the set of all subwords of words in $\mathcal{L}$. E.g. $\text{SubWords}(abc) = \{ abc, bc, ac, ab, a, b, c, \epsilon \}$

**Theorem** (Haines 1969) For all $\mathcal{L} \subseteq \Sigma^*$, $\downarrow(\mathcal{L})$ is regular.
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Regular representations of downward closures are very useful:
- Regular languages are well behaved under many transformations.
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Computing downwards closure of higher-order pushdown languages

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- Many systems permit synchronisation with a regular language.

Example: In message-passing concurrency, complex environments can be abstracted by the downward closure of the messages it sends (or processes it spawns).
Theorem (Hague, Kochems & O. POPL16)

The downward closure of every order-$n$ pushdown language is computable.

(Zetzsche 2015) If $C$ is a full trio and has decidable $\text{Diagonal}(C)$, then it has computable downward closures.

Fix $\Sigma = \{ a_1, \ldots, a_n \}$. $\text{Diagonal}(C)$: Given $L \in C$, does it hold that

$$\forall k \geq 0 . \exists w_k \in L . \left( \#_{a_1}(w_k) \geq k \land \cdots \land \#_{a_n}(w_k) \geq k \right)$$
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Several Consequences

1. Reachability for parameterised concurrent systems of HOPDA communicating asynchronously via a shared global register (La Torre et al. 2015)

2. Finiteness of a language defined by a HOPDA, and

3. Downward closure of the Parikh image of a HOPDA.
For $n \geq 0$, let $\text{RecSchTree}_n$ be the class of $\Sigma$-labelled trees generated by order-$n$ recursion schemes.

### Some Properties

1. **Hierarchy Theorem** (Damm 1982) for $\langle \text{RecSchTree}_n \mid n \in \omega \rangle$

2. The hierarchy is highly expressive: order-0 are the regular trees, order-1 are the algebraic trees (Courcelle 1995); order-2 are the hyperalgebraic trees (Knapik et al. 2001).

3. Machine characterization: order-$n$ trees are exactly those generated by order-$n$ collapsible pushdown automata (HMOS LiCS 2008)

4. **MSO theories are decidable** (to date, the “largest” known such hierarchy of trees).
Outline

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3. Some Open Problems
Like standard model checking, higher-order model checking is mostly a whole program analysis. This can seem surprising: higher order is supposed to aid modular structuring of programs!
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**Aim:** model check the computation trees of higher-type functional programs (＝ Böhm trees i.e. trees with binders).
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Need a denotational model to support compositional model checking, which should be strategy aware (i.e. modelling Böhm trees, and witnesses of correctness properties of Böhm trees), and organisable into a cartesian closed category of parity games.
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Unfortunately the elegant theorems of “Rabin’s Heaven” fail in the world of Böhm trees.
Theorems of “Rabin’s Heaven” do not hold for Böhm trees

Let $\Gamma = a : o, \ b : o \to ((o \to o) \to o) \to o$ and

\[
\Gamma \vdash Y (\lambda f. \lambda y^o. \lambda x^{o \to o}. b(x\ y) (f(x\ y))) \ a : (o \to o) \to o.
\]

$BT(M)$
- uses infinitely many variable names,
and each variable occurs infinitely often.
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$BT(M)$ 
- uses infinitely many variable names, and each variable occurs infinitely often.
- has an undecidable MSO theory! (Clairambault & Murawski FSTTCS’13)
An expressive yet decidable logic for higher-type Böhm trees?

Böhm trees: concrete repr. of higher-order functions on infinite trees.

Take tree property \( \mathcal{P} \) := “There are only finitely many occurrences of bound variables in each branch.”
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Take tree property \( \mathcal{P} := \text{"There are only finitely many occurrences of bound variables in each branch."} \)

Questions: Automata-Logic-Games Correspondence for Higher-Type Trees

1. Is there an expressive logic \( \mathcal{L} \) that can describe properties such as \( \mathcal{P} \)?
2. Is there a class of automata equi-expressive with \( \mathcal{L} \)?
3. What kind of games characterise the acceptance problem?
4. Is \( \mathcal{L} \) decidable for definable Böhm trees?
## Higher-type Automata-Logic-Games Correspondence

<table>
<thead>
<tr>
<th>$\Sigma$-labelled trees</th>
<th>Higher-type Böhm trees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternating Parity Tree Automata</td>
<td>Alternating Dependency Tree Automata</td>
</tr>
<tr>
<td>- has rules that read $\lambda$-binders</td>
<td>- generalise Stirling’s ADTA to $\omega$-regular winning condition</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mu-calculus</th>
<th>Higher-type Mu-Calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi ::= P \mid \neg \varphi \mid \varphi \lor \psi \mid \nu\alpha.\varphi \mid [i] \varphi$</td>
<td>- $\varphi ::= \forall x \mid \lambda x.\psi \mid \cdots$</td>
</tr>
<tr>
<td>- $\varphi$ detects variables; $\lambda x.$- detects $\lambda$-binding</td>
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</table>

| Parity Game | Type-Checking Game $\models u : \alpha$ |

Further, checking these properties against $\lambda \rightarrow \Upsilon$-definable Böhm trees is decidable. (Tsukada & O. LICS14)
Outline

1 Higher-Order Pushdown Automata: A Model of Higher-order Computation
   - Properties of the Maslov(= Higher-order Pushdown) Hierarchy of Word Languages
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3 Some Open Problems
1. **Equivalence of Recursion Schemes** asks whether two given recursion schemes generate the same tree. (Recursively equivalent to Böhm Tree Equivalence of $\lambda Y$-terms.) Is the problem decidable?

2. The **Nondeterministic Safety Conjecture**: there is a word language recognisable by a nondeterministic $n$-CPDA, but not by any nondeterministic HOPDA. False for $n = 2$; open for $n \geq 3$.

3. **Are Unsafe Word Languages Context Sensitive?** Answer is Yes for order up to 3 (Kobayashi et al. FoSSaCS14).

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5. **Extensions of Higher-Order Model Checking**

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