Higher-Order Model Checking

A Tutorial

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Model Checking

The development of techniques (notably model checking) for the computer-aided verification of computing systems has been a truly successful application of logic to computer science.

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What is Model Checking?

Problem: Given a system Sys (e.g. an OS) and a correctness property $Spec$ (e.g. deadlock freedom), does Sys satisfy $Spec$?

The model checking approach:

1. Find an abstract model M of the system Sys.
2. Describe the property $Spec$ as a formula φ of a (decidable) logic.
3. Exhaustively check if φ is violated by M.

Verification of Higher-Order Programs

In the past two decades, there have been significant advances in the theory and engineering of scalable software model checkers (especially for first-order imperative programs such as C). E.g. SLAM, BLAST, CMBC.

- These techniques are much less useful for higher-order programs.
- Yet higher-order features (e.g. lambdas, streams) are already standard in today's leading languages: Java8, C++11, C#5.0, Python, Scala, etc.

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Verifying higher-order functional programs: 2 standard approaches

1. Type-based program analysis. E.g. type-and-effect, qualifier, linear
   - sound, scalable but often imprecise
2. Theorem proving and dependent types. E.g. Coq, Agda
   - accurate, typically requires human intervention; does not scale well

We present an approach to verifying higher-order programs via higher-order model checking.

Tutorial Outline

Higher-Order Model Checking is the model checking of infinite structures, such as trees, that are defined by recursion schemes (equivalently $\lambda\mathbf{Y}$-calculus) and related families of higher-order generators.

This tutorial has four parts:

1. Introduction (Ong)
2. Applications to Program Verification (Kobayashi)
3. Type Systems and Algorithms for Higher-Order Model Checking (Kobayashi)
4. Advanced Topics (Ong)

Simple Types (Church JSL 1940)

Types $\quad A \quad ::= \quad o \quad | \quad (A \to B)$

o is the type of trees.

Order of a type: measures "nestedness" on LHS of $\to$.

$$\begin{array}{rcl} \mathsf{order}(o) & := & 0 \\ \mathsf{order}(A \to B) & := & \max(\mathsf{order}(A) + 1, \mathsf{order}(B)) \end{array}$$

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Examples

1. $\mathbb{N} \to \mathbb{N}$ and $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ both have order 1;
2. $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ has order 2.

Notation $\quad e : A \quad$ means “expression e has type A”.

(Park 68, de Roever 72, Nivat 72, Nivat-Courcelle 78, Damm 82, . . .)

HORS are grammars for trees (and tree languages).

Order-n recursion schemes over Σ = programs of the order-n fragment of $\lambda^{\rightarrow}$**Y**-calculus (i.e. simply-typed λ-calculus + **Y** + order-1 Σ-symbols).

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Concretely, a HORS is a finite set of simply-typed (higher-order) functions, defined by mutual recursion over Σ, with a distinguished start function S of ground type.

Example (order 1). $\Sigma = \{\, f : o \to (o \to o),\ g : o \to o,\ a : o \,\}$.

$$\mathcal{G} \ : \left\{ \begin{array}{rcl} S & \to & F\,a \\ F\,x & \to & (f\,x)\,(F\,(g\,x)) \end{array} \right.$$

Example (order 1)

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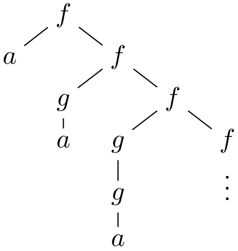


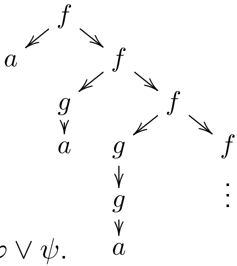
The tree generated, $[\![\, \mathcal{G} \,]\!]$, is the abstract syntax tree underlying $(f\,a)\,(f\,(g\,a)\,(f\,(g\,(g\,a))(\cdots)))$.

Many equivalent ways of defining $[\![\, \mathcal{G} \,]\!]$ (as least fixpoint, least solution, initial algebra, etc.).

A Basic Verification Problem in Higher-Order Computation

E.g. Consider properties of nodes of $[\![G]\!]$:

- $\varphi =$ "Infinitely many f-nodes are reachable".
- $\psi =$ "Only finitely many g-nodes are reachable".



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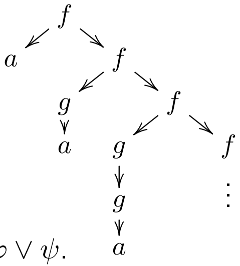

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MSO Model-Checking Problem for Trees generated by HORS

- INSTANCE: An order-n recursion scheme $\mathcal{G}$, and an MSO formula φ
- QUESTION: Does the Σ-labelled tree $[\![\mathcal{G}]\!]$ satisfy φ?

QUESTION: Is the above problem decidable?

Theorem (O. LICS06)

For $n \geq 0$, the alternating parity tree automaton (APT) model-checking problem for order-n recursion schemes is n-EXPTIME complete. Hence the MSO model checking problem is decidable.

Decidability and Complexity of MSO Model Checking of HORS

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Proofs of Decidability of HOMC / Models of Higher-Order Computation

1. Game semantics (O. LICS06)
2. Collapsible pushdown automata (Hague, Murawski, O. & Serre LICS08)
3. Intersection types (Kobayashi & O. LICS09)
4. Krivine machine (Salvati & Walukiewicz ICALP11)

Outline of Part 4

1. Higher-Order Pushdown Automata: A Model of Higher-order Computation
   - Properties of the Maslov(= Higher-order Pushdown) Hierarchy of Word Languages
   - Computing Downwards Closure of Higher-order Pushdown Languages
2. Model Checking Higher-type Böhm Trees
   - Challenge of Compositional Higher-order Model Checking
   - Automata-Logic-Games Correspondence for Higher-type Computation
3. Some Open Problems

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Operations on 2-stacks: s_i ranges over 1-stacks.

$$
\begin{array}{rcll}
push_2 & : & [s_1 \cdots s_{i-1}\, \underbrace{[\gamma_1 \cdots \gamma_n]}_{s_i}] & \mapsto \quad [s_1 \cdots s_{i-1}\, s_i\, s_i] \\
pop_2 & : & [s_1 \cdots s_{i-1}\, [\gamma_1 \cdots \gamma_n]] & \mapsto \quad [s_1 \cdots s_{i-1}] \\
push_1\, \gamma & : & [s_1 \cdots s_{i-1}\, [\gamma_1 \cdots \gamma_n]] & \mapsto \quad [s_1 \cdots s_{i-1}\, [\gamma_1 \cdots \gamma_n\, \gamma]] \\
pop_1 & : & [s_1 \cdots s_{i-1}\, [\gamma_1 \cdots \gamma_n\, \gamma_{n+1}]] & \mapsto \quad [s_1 \cdots s_{i-1}\, [\gamma_1 \cdots \gamma_n]]
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Higher-order pushdown automata (HOPDA) [Maslov 74]

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\end{array}
$$

Idea extends to all finite orders: an order-n PDA has an order-n stack, and has $push_i$ and pop_i for each $1 \leq i \leq n$.

Example: $L := \{ a^n \, b^n \, c^n : n \geq 0 \}$ is recognisable by an order-2 PDA

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$$
\begin{tikzcd}
q_1\,[[\,]] \arrow[r, "a"] & q_1\,[[\,]\,[z]] \arrow[r, "a"] & q_1\,[[\,]\,[z]\,[z\ z]] \arrow[d, "b"] \\
 & & q_2\,[[\,]\,[z]\,[z]] \arrow[d, "b"] \\
q_3\,[[\,]] & q_3\,[[\,]\,[z]] \arrow[l, "c"'] & q_2\,[[\,]\,[z]\,[\,]] \arrow[l, "c"']
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$$
\begin{tikzcd}[column sep=huge]
q_1 \arrow[loop above, "- \xrightarrow{a} push_2 \,;\, push_1 z"] \arrow[r, "z \xrightarrow{b} pop_1 z"'] & q_2 \arrow[loop above, "z \xrightarrow{b} pop_1"] \arrow[r, "\perp \xrightarrow{c} pop_2"'] & q_3 \arrow[loop above, "z \xrightarrow{c} pop_2"]
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'read a' 'read b' 'read c'

Some properties of the Maslov Hierarchy (Maslov 74, 76)

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A recent breakthrough

Theorem (Inaba + Maneth FSTTCS08)

All languages of the Maslov Hierarchy are context-sensitive. So Maslov Hierarchy refines Chomsky Hierarchy.

Maslov – a robust hierarchy of word languages

Theorem (Equi-expressivity)

For each $n \geq 0$, the three formalisms

1. *order-n pushdown automata (Maslov 76)*
2. *order-n safe recursion schemes (Damm 82, Damm + Goerdt 86)*
3. *order-n indexed grammars (Maslov 76)*

generate the same class of word languages.

Computing downwards closure of higher-order pushdown languages

Downward closure of $\mathcal{L}$, $\downarrow(\mathcal{L})$, is the set of all subwords of words in $\mathcal{L}$.
E.g. $SubWords(a\,b\,c) = \{\,a\,b\,c,\ b\,c,\ a\,c,\ a\,b,\ a,\ b,\ c,\ \epsilon\,\}$

Theorem (Haines 1969) For all $\mathcal{L} \subseteq \Sigma^*$, $\downarrow(\mathcal{L})$ is regular.

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Example: In message-passing concurrency, complex environments can be abstracted by the downward closure of the messages it sends (or processes it spawns).

Theorem (Hague, Kochems & O. POPL16)

The downward closure of every order-n pushdown language is computable.

(Zetzsche 2015) If $\mathcal{C}$ is a full trio and has decidable $\textsc{Diagonal}(\mathcal{C})$, then it has computable downward closures.

Fix $\Sigma = \{ a_1, \cdots, a_n \}$. $\textsc{Diagonal}(\mathcal{C})$: Given $L \in \mathcal{C}$, does it hold that $\forall k \geq 0 \, . \, \exists w_k \in L \, . \, \big(\#_{a_1}(w_k) \geq k \wedge \cdots \wedge \#_{a_n}(w_k) \geq k\big)$?

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Several Consequences

1. Reachability for parameterised concurrent systems of HOPDA communicating asynchronously via a shared global register (La Torre et al. 2015)
2. Finiteness of a language defined by a HOPDA, and
3. Downward closure of the Parikh image of a HOPDA.

Hierarchy of trees generated by HORS

For $n \geq 0$, let $\textbf{RecSchTree}_n$ be the class of Σ-labelled trees generated by order-n recursion schemes.

Some Properties

1. Hierarchy Theorem (Damm 1982) for $\langle\, \textbf{RecSchTree}_n \mid n \in \omega \,\rangle$
2. The hierarchy is highly expressive: order-0 are the regular trees, order-1 are the algebraic trees (Courcelle 1995); order-2 are the hyperalgebraic trees (Knapik et al. 2001).
3. Machine characterization: order-n trees are exactly those generated by order-n collapsible pushdown automata (HMOS LiCS 2008)
4. MSO theories are decidable (to date, the "largest" known such hierarchy of trees).

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Compositional Higher-Order Model Checking? ... Several Obstacles

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4. Unfortunately the elegant theorems of "Rabin's Heaven" fail in the world of Böhm trees.

Theorems of "Rabin's Heaven" do not hold for Böhm trees

Let $\Gamma = a : o,\ b : o \to ((o \to o) \to o) \to o$ and

$$\Gamma \ \vdash\ \underbrace{\mathbf{Y}\,(\lambda f.\lambda y^{o}.\lambda x^{o \to o}.b\,(x\,y)\,(f\,(x\,y)))\,a}_{M} \ :\ (o \to o) \to o.$$

$\mathrm{BT}(M)$

- uses infinitely many variable names,
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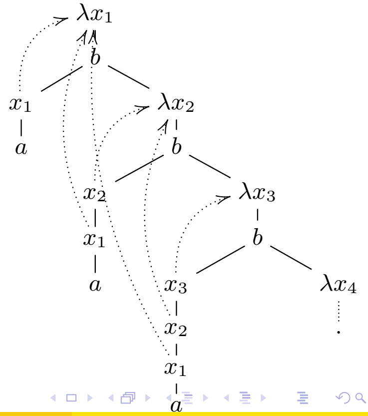

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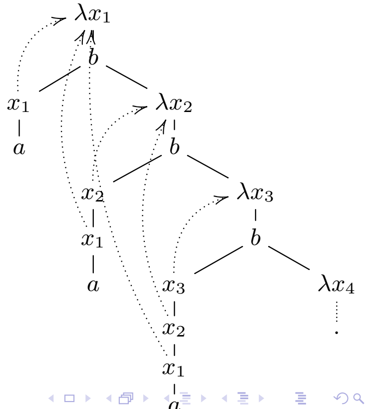
- uses infinitely many variable names, and each variable occurs infinitely often.

- has an undecidable MSO theory! (Clairambault & Murawski FSTTCS’13)



An expressive yet decidable logic for higher-type Böhm trees?

Böhm trees: concrete repr. of higher-order functions on infinite trees.

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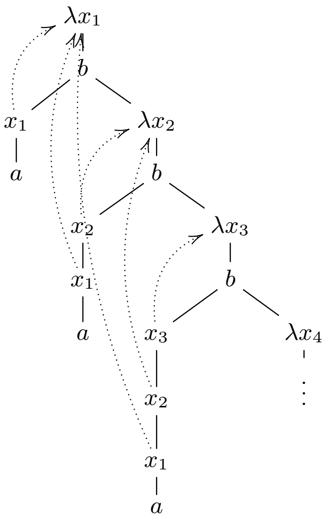

Questions: Automata-Logic-Games Correspondence for Higher-Type Trees

1. Is there an expressive logic $\mathcal{L}$ that can describe properties such as $\mathfrak{P}$?
2. Is there a class of automata equi-expressive with $\mathcal{L}$?
3. What kind of games characterise the acceptance problem?
4. Is $\mathcal{L}$ decidable for definable Böhm trees?

Σ**-labelled trees**	**Higher-type Böhm trees**
Alternating Parity Tree Automata	Alternating Dependency Tree Automata - has rules that read λ-binders - generalise Stirling's ADTA to ω-regular winning condition
Mu-calculus $\varphi ::= P \mid \neg\varphi \mid \varphi \vee \psi$ $\mid \nu\alpha.\varphi \mid [i]\varphi$	Higher-type Mu-Calculus - $\varphi ::= \mathbb{v}_x \mid \mathbb{\lambda} x.\psi \mid \cdots$ - $\mathbb{v}_x$ detects variables; $\mathbb{\lambda} x.$- detects λ-binding
Parity Game	Type-Checking Game $\models u : \alpha$

Further, checking these properties against $\lambda^{\rightarrow}\mathbf{Y}$-definable Böhm trees is decidable. (Tsukada & O. LICS14)

Outline

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